Name: ....

Gosford High School

2023 Trial HSC examination

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/ or calculations

Total Marks

70

Section I – 10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Detach the *Multiple-choice answer sheet* from the last page of this question booklet.

Section II – 60 marks

- Attempt Questions 11-14
- Allow about 1 hours and 45 minutes for this section

Section I

(10 marks)
Attempt Questions 1 – 10
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 10.

1 Which expression is equal to $\int \sin^2(3x)\,dx$

A. $\frac{1}{2}\left(x-\frac{1}{3}\sin(3x)\right)+C$

B. $\frac{1}{2}\left(x+\frac{1}{3}\sin(3x)\right)+C$

C. $\frac{1}{2}\left(x-\frac{1}{6}\sin(6x)\right)+C$

D. $\frac{1}{2}\left(x+\frac{1}{6}\sin(6x)\right)+C$

2 Which of the following is the solution to $\frac{2}{x-2}<2$?

A. $x<2$ or $x>3$

B. $2<x<3$

C. $-2<x<3$

D. $-3<x<2$

3 If $\sin x=\frac{3}{5}$ and $\frac{\pi}{2}\le x\le\pi$, evaluate $\tan 2x$.

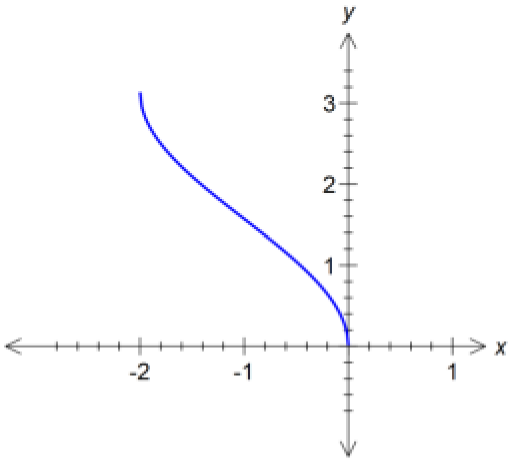
A. $\frac{-7}{24}$

B. $\frac{-24}{7}$

C. $\frac{7}{24}$

D. $\frac{24}{7}$

4 Which of the following best describes this function?



A. $y = \sin^{-1}(x+1)$

B. $y = \sin^{-1}(x)+1$

C. $y = \cos^{-1}(x+1)$

D. $y = \cos^{-1}(x)+1$

5 In how many ways can 5 people be selected from a group of 6 and then arranged in a line so that the two oldest people in the selected group are at either end of the line? (Note, no two people are the same age)

A. 720

B. 144

C. 72

D. 36

6 The temperature T^{o} C of water in a jug is given by $T = 20 + 80e^{-0.2t}$. What is the rate at which the water is cooling when its temperature has fallen to half its initial value?

A. 6^{o} C per minute

B. 8^{o} C per minute

C. 12^{o} C per minute

D. 16^{o} C per minute

7 What is the range of the function $f(x) = \tan^{-1}(\sin x)$

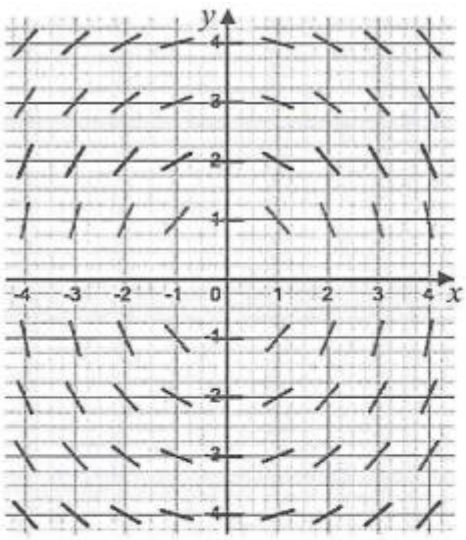
A. $\left(\dfrac{-\pi}{4}, \dfrac{\pi}{4}\right)$

B. $\left[\dfrac{-\pi}{4}, \dfrac{\pi}{4}\right]$

C. $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$

D. $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$

8 Which differential equation is represented by the following slope field?



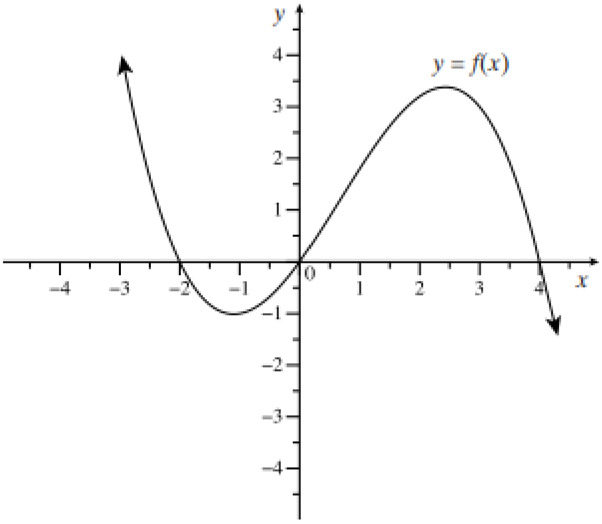
A. $\dfrac{dy}{dx} = \dfrac{-x}{y}$

B. $\dfrac{dy}{dx} = \dfrac{-y}{x}$

C. $\dfrac{dy}{dx} = \dfrac{x}{y}$

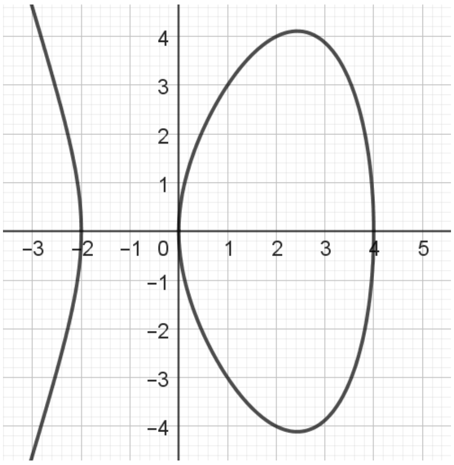
D. $\dfrac{dy}{dx} = \dfrac{y}{x}$

9 The graph of $y = f(x)$ is shown below.

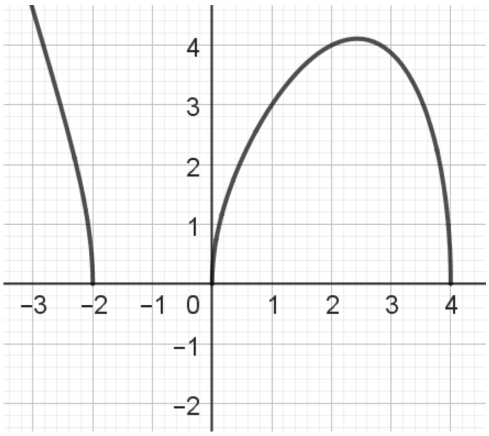


The graph of $y = \dfrac{1}{f(x)}$ is best represented by:

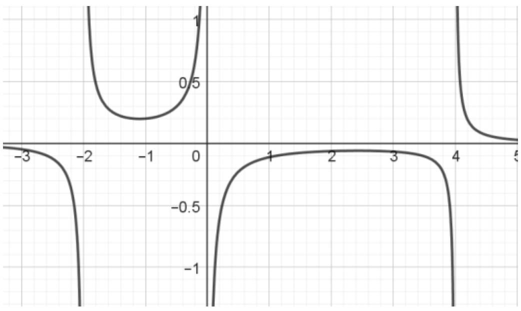
A



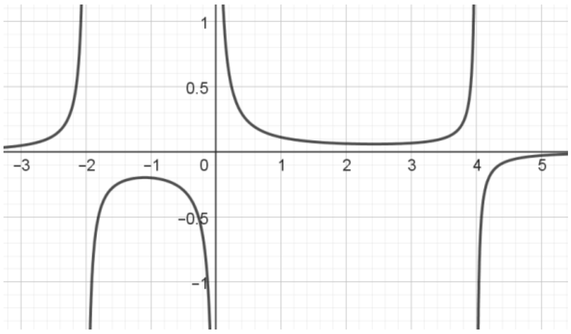
B



C



D



10 Which of the following is equal to $\cos 3\alpha \sin 2\alpha$?

A. $\frac{1}{2}(\sin 5\alpha + \sin \alpha)$

B. $\frac{1}{2}(\sin 5\alpha - \sin \alpha)$

C. $\frac{1}{2}\left(\sin \frac{5\alpha}{2} + \sin \frac{\alpha}{2}\right)$

D. $\frac{1}{2}\left(\sin \frac{5\alpha}{2} - \sin \frac{\alpha}{2}\right)$

End of Section I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section.

Answer each question in a separate booklet.

Question 11 (15 marks) Start a new booklet.

a) The polynomial $P(x) = 2x^3 - 8x^2 + 7x - 14$ has roots α, $-\alpha$ and β. 1

What is the value of β ?

b) Use the $t = \tan\frac{x}{2}$ results to show that $\cot x + \tan\frac{x}{2} = \operatorname{cosec} x$ 2

c) $\int \frac{2}{x^2+4}\,dx$ 2

d) Find the constant term in the expansion $\left(2x + \frac{3}{x^3}\right)^8$ 3

e) OAB is a triangle where $\overrightarrow{OA} = 2\underset{\sim}{i} - k\underset{\sim}{j}$, $\overrightarrow{OB} = k\underset{\sim}{i} + 4\underset{\sim}{j}$ and $AB = 5\sqrt{2}$

i) Find $\overrightarrow{AB}$ in terms of *k*. 1

ii) Find any values of *k*. 3

f) Show that the derivative of $\frac{\cos^{-1}(2x)}{x}$ is equal to $\frac{-\left[2x + \cos^{-1}(2x)\sqrt{1-4x^2}\right]}{x^2\sqrt{1-4x^2}}$ 3

Question 12 (15 marks) Start a new booklet.

a) Evaluate $\int_0^1 x^2\sqrt{1+3x^3}\,dx$ using the substitution $u=1+3x^3$ 3

b) Given α, β, γ are the roots of the equation $x^3-3x^2-x+3=0$, evaluate $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$. 3

c) i) Express $\sqrt{3}\,\text{Cos}\,\theta-\text{Sin}\,\theta$ in the form $R\,\text{Cos}(\theta+\alpha)$ where $0\le\alpha\le\frac{\pi}{2}$. 2

ii) Hence, or otherwise, solve $\sqrt{3}\,\text{Cos}\,\theta-\text{Sin}\,\theta=1$ for $0\le\theta\le2\pi$ 2

d) If $\underset{\sim}{a}=-6\underset{\sim}{i}-2\underset{\sim}{j}$ and $\underset{\sim}{b}=\underset{\sim}{i}+2\underset{\sim}{j}$

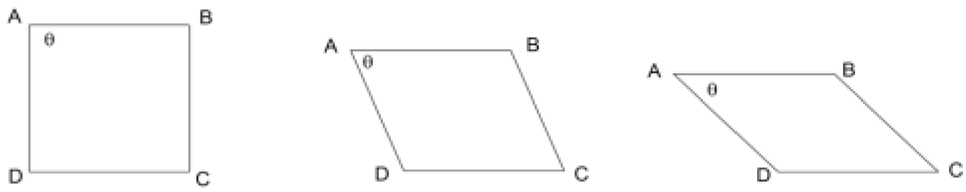
i) Show that $\underset{\sim}{a}\circ\underset{\sim}{b}=-10$ 1

ii) Hence find the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ 1

e) Use mathematical Induction to prove that $11^{2n}+11^n+8$ is a multiple of 10 for $n\ge1$. 3

Question 13 (15 marks) Start a new page.

a)



A square *ABCD* of side length 1 unit is "gradually" pushed over to become a rhombus. The angle at A θ decreases at a constant rate of 0.1 radian per second?

i) At what rate is the area of the rhombus *ABCD* decreasing when $\theta = \frac{\pi}{6}$? 3

ii) At what rate is the shorter diagonal of the rhombus *ABCD* decreasing when $\theta = \frac{\pi}{3}$? 3

b) Show that the solution to the differential equation $\frac{dy}{dx} = 1 + \frac{2y}{3}$, given $y = -1$ when $x = 2$ can be written in the form $y = \frac{e^{\frac{2(x-2)}{3}} - 3}{2}$. 4

c) A rabbit population grows according to the logistic equation $\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{10000}\right)$, where P is the number of rabbits after t months. The initial population is 1000.

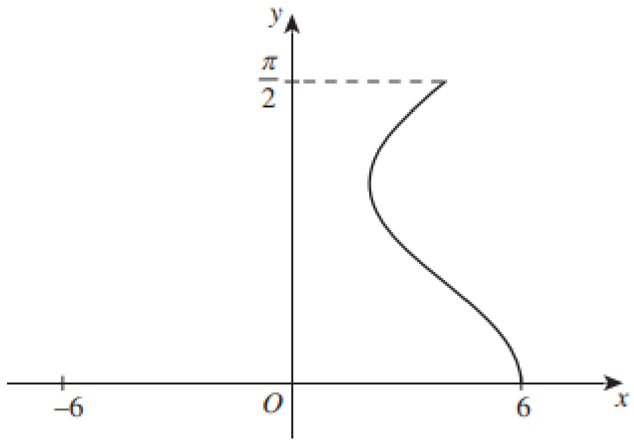
i) Show that $\frac{10000}{P(10000-P)} = \frac{1}{P} + \frac{1}{10000-P}$. 1

ii) Find the population of rabbits after seven months. 4

Question 14 (15 marks) Start a new booklet

a) The diagram shows the region bounded by the y-axis and $x = 4\cos(3y) + 2$, where $0 \le y \le \frac{\pi}{2}$. 3

The region is rotated about the y-axis to form a solid.



Find the exact volume, *V*, of the solid formed.

b) A particle, A, is projected from the origin with an initial velocity of $16\underset{\sim}{i} + 30\underset{\sim}{j}\ \text{ms}^{-1}$.

At the same time, particle B is projected towards the origin from a point that is 60 m to the right of the origin and 25 m above the origin with an initial velocity of $-8\underset{\sim}{i} + 20\underset{\sim}{j}\ \text{ms}^{-1}$.

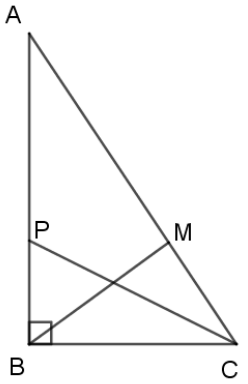
Let the acceleration due to gravity be 9.8ms^{-2}.

The position vector of particle A at time t seconds is given by $A(t) = 16t\underset{\sim}{i} + (30t - 4.9t^2)\underset{\sim}{j}$.
(Do NOT prove this.)

i) Find the position vector of particle B (i.e. $B(t)$) at time *t* seconds. 3

ii) Find the time and point at which the two particles collide. 3

c) In the figure $A\hat{B}C = 90°$, $PA = 2 \times BP$, $AM = 2 \times MC$.



Let $\overrightarrow{BA} = \underset{\sim}{v}$ and $\overrightarrow{BC} = \underset{\sim}{u}$.

i) Express $\overrightarrow{BM}$ and $\overrightarrow{CP}$ in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$. 3

ii) If $BM \perp CP$, find $\dfrac{|BA|}{|BC|}$. 3

End of the Examination. ☺

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Gosford High School

2023 Trial HSC examination

Mathematics Extension 1

Solutions

2023 Year 12 HSC Advanced Examination

Section 1: Solutions

1.	A ¡	B ¡	C |	D ¡
2.	A |	B ¡	C ¡	D ¡
3.	A ¡	B |	C ¡	D ¡
4.	A ¡	B ¡	C |	D ¡
5.	A ¡	B ¡	C |	D ¡
6.	A |	B ¡	C ¡	D ¡
7.	A ¡	B |	C ¡	D ¡
8.	A |	B ¡	C ¡	D ¡
9.	A ¡	B ¡	C ¡	D |
10.	A ¡	B |	C ¡	D ¡

Section I

1 Which expression is equal to $\int \sin^2(3x)\,dx$

C. $\frac{1}{2}\left(x-\frac{1}{6}\sin(6x)\right)+C$

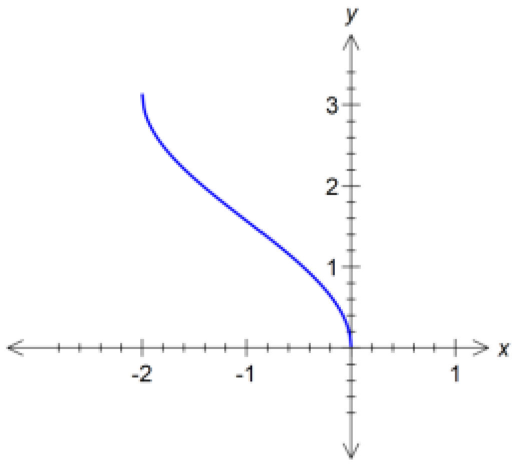
2 Which of the following is the solution to $\frac{2}{x-2}<2$?

A. $x<2$ or $x>3$

3 If $\sin x=\frac{3}{5}$ and $\frac{\pi}{2}\le x\le \pi$, evaluate $\tan 2x$.

B. $\frac{-24}{7}$

4 Which of the following best describes this function?



C. $y=\cos^{-1}(x+1)$

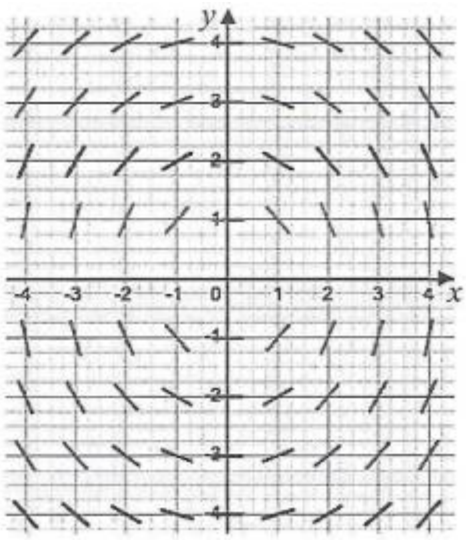
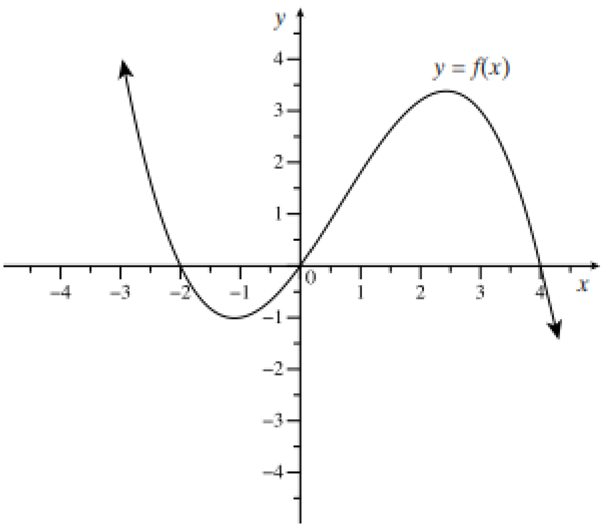
5 In how many ways can 5 people be selected from a group of 6 and then arranged in a line so that the two oldest people in the selected group are at either end of the line? (Note, no two people are the same age)

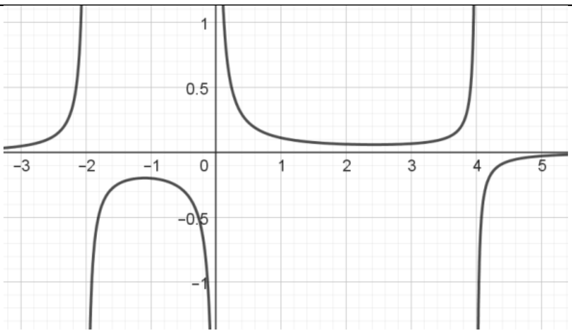
C. 72

6 The temperature T^{o} C of water in a jug is given by $T=20+80e^{-0.2t}$. What is the rate at which the water is cooling when its temperature has fallen to half its initial value?

A. 6° C per minute

7 What is the range of the function $f(x)=\tan^{-1}(\sin x)$

	B. $\left[\frac{-\pi}{4},\frac{\pi}{4}\right]$
8	Which differential equation is represented by the following slope field?  A. $\frac{dy}{dx}=\frac{-x}{y}$
9	The graph of $y=f(x)$ is shown below.  The graph of $y=\frac{1}{f(x)}$ is best represented by:

	D. 
10	Which of the following is equal to $\cos 3\alpha \sin 2\alpha$? B. $\frac{1}{2}(\sin 5\alpha - \sin\alpha)$

Mathematics Extension 1

Section II Solutions

Question 11

a) The polynomial $P(x)=2x^3-8x^2+7x-14$ has roots α, $-\alpha$ and β. What is the value of β ?

Solution:

$$\alpha-\alpha+\beta = -\frac{(-8)}{2}$$

$$\therefore \beta = 4$$

Marks	Guideline
1	For correct answer
Marker's Comment	
	Question done well overall

b) Use the $t=\tan\frac{x}{2}$ results to show that $\cot x+\tan\frac{x}{2}=\operatorname{cosec} x$

Solution:

$$\begin{aligned}\text{LHS} &= \cot x+\tan\left(\frac{x}{2}\right)\\ &= \frac{1-t^2}{2t}+t\\ &= \frac{1-t^2+2t^2}{2t}\\ &= \frac{1+t^2}{2t}\\ &= \operatorname{cosec} x\\ &= \text{RHS}\end{aligned}$$

Marks	Guideline
2	Correct solution
1	For correct substitution and making progress towards solution.
Marker's Comment	Students should take care to only work on one side of the equation at a time. Define the LHS and RHS separately and work on them individually.

c) $\displaystyle\int \frac{2}{x^2+4}\,dx$

Solution:

$$\begin{aligned}\int \frac{2}{x^2+4}\,dx &= 2\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+c\\ &= \tan^{-1}\left(\frac{x}{2}\right)+c\end{aligned}$$

Marks	Guideline
2	Correct solution
1	For using inverse tan or getting coefficient of 1.
Marker's Comment	Question overall done well.

d)	Find the constant term in the expansion $\left(2x+\frac{3}{x^3}\right)^8$

Solution:

Term in x^k $=\binom{8}{k}(2x)^k\left(3x^{-3}\right)^{8-k}$

$=\binom{8}{k}2^k\times x^k\times 3^{8-k}\times x^{3k-24}$

$=\binom{8}{k}2^k\times 3^{8-k}\times x^{4k-24}$

For constant term $4k-24=0 \Rightarrow k=6$

The constant term is $\binom{8}{6}\times 2^6\times 3^{8-6}=16128$

Marks	Guideline
3	Correct solution
2	Correct value for *k* or correct answer from their *k*.
1	Correct substitution or showing some progress.
Marker's Comment	Question overall done well.

e)	OAB is a triangle where $\overrightarrow{OA}=2\underset{\sim}{i}-k\underset{\sim}{j}$, $\overrightarrow{OB}=k\underset{\sim}{i}+4\underset{\sim}{j}$ and $AB=5\sqrt{2}$
i)	Find $\overrightarrow{AB}$ in terms of k.

Solution:

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$=k\underset{\sim}{i}+4\underset{\sim}{j}-\left(2\underset{\sim}{i}-k\underset{\sim}{j}\right)$

$=(k-2)\underset{\sim}{i}+(k+4)\underset{\sim}{j}$

Marks	Guideline
1	For correct answer
Marker's Comment	Question done well overall.

ii)	Find any values of k.

Solution:

$(k-2)^2+(4+k)^2=50$

$\therefore k^2-4k+4+k^2+8k+16=50$

$\therefore k^2+2k-15=0$

$\therefore k=3,-5$

Marks	Guideline
3	Correct solution
2	Correctly solves their quadratic or gets one of the correct values.
1	Showing some progress.
Marker's Comment	This question was generally done well if part i was correct.

f)	Show that the derivative of $\frac{\cos^{-1}(2x)}{x}$ is equal to $\frac{-\left[2x+\cos^{-1}(2x)\sqrt{1-4x^2}\right]}{x^2\sqrt{1-4x^2}}$

Solution:

$$\frac{d}{dx}\left(\frac{\cos^{-1}(2x)}{x}\right)=\frac{x\left(\frac{-2}{\sqrt{1-4x^2}}\right)-\cos^{-1}(2x)\times 1}{x^2}$$

$$=\frac{\frac{-2x}{\sqrt{1-4x^2}}-\cos^{-1}(2x)}{x^2}$$

$$=\frac{-\left[2x+\cos^{-1}(2x)\sqrt{1-4x^2}\right]}{x^2\sqrt{1-4x^2}}$$

Marks	Guideline
3	Correct solution
2	Correct substitution into Quotient Rule.
1	Evidence of Quotient rule or correctly differentiates inverse cos.
Marker's Comment	Quotient rule was applied quite well and consistently.

Question 12

a) Evaluate $\int_0^1 x^2\sqrt{1+3x^3}\,dx$ using the substitution $u=1+3x^3$

Solution:

$u=1+3x^3 \Rightarrow \frac{du}{dx}=9x^2$

$\therefore \frac{1}{9}du = x^2dx$

At $x=0,\ u=1$

$x=1,\ u=4$

$$\int_0^1 x^2\sqrt{1+3x^3}\,dx = \frac{1}{9}\int_1^4 \sqrt{u}\,du$$

$$=\frac{1}{9}\left[\frac{2}{3}u^{3/2}\right]$$

$$=\frac{2}{27}[8-1]$$

$$=\frac{14}{27}$$

Marks	Guideline
3	For correct solution
2	Correct integral from substitution or Correct value from their substitution/boundary values. Or equivalent merit.
1	Correct boundary values for *u* or correct derivative of substitution or equivalent merit.
Marker's Comment	Mostly well done. Main errors occurred when students did not find the bounds for u. Another concerningly common mistake was not reading the given u substitution correctly and writing $u = 1+3x^2$ making the derivative $u' = 6x$.

b) Given α, β, γ are the roots of the equation $x^3-3x^2-x+3=0$, evaluate $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$

Solution:

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2}{\alpha^2\beta^2\gamma^2}$$

$$\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2 = (\alpha\beta+\alpha\gamma+\beta\gamma)^2-2\left(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2\right)$$

$$=(\alpha\beta+\alpha\gamma+\beta\gamma)^2-2\left(\alpha\beta\gamma(\alpha+\beta+\gamma)\right)$$

$$=(-1)^2-2(-3\times 3)$$

$$=19$$

$$\alpha^2\beta^2\gamma^2 = (-3)^2$$

$$=9$$

$$\therefore \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{19}{9}$$

Marks	Guideline
3	Correct solution
2	Correct simplification of sum of the square of the roots, plus 1 correct from 1 Mark or equivalent merit.
1	Correct value for either sum of roots one at a time, two at a time or three at a time.
Marker's Comment	Poorly done by most. Better responses were able to successfully simplify

			the fraction and correctly found the values for the sums and products of the roots.

ci) Express $\sqrt{3}\cos\theta - \sin\theta$ in the form $R\cos(\theta+\alpha)$ where $0 \le \alpha \le \frac{\pi}{2}$

Solution:

$$\sqrt{3}\cos\theta - \sin\theta = R\cos(\theta+\alpha)$$
$$= R\sin\alpha\cos\theta - R\cos\alpha\sin\theta$$

$$\therefore\ R\sin\alpha = \sqrt{3}$$
$$R\cos\alpha = 1$$

$$\therefore \tan\alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{6}$$

$$R = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$$

$$\therefore \sqrt{3}\cos\theta - \sin\theta = 2\cos\left(\theta + \frac{\pi}{6}\right)$$

Marks	Guideline
2	Correct solution
1	Correct value for either R or θ
Marker's Comment	Well done overall.

cii) Hence, or otherwise, solve $\sqrt{3}\cos\theta - \sin\theta = 1$ for $0 \le \theta \le 2\pi$

Solution:

$$2\cos\left(\theta+\frac{\pi}{6}\right) = 1 \qquad 0 \le \theta \le 2\pi$$

$$\therefore \cos\left(\theta+\frac{\pi}{6}\right) = \frac{1}{2} \qquad 0 \le \theta + \frac{\pi}{6} \le \frac{13\pi}{6}$$

$$\therefore \theta + \frac{\pi}{6} = \frac{\pi}{3} \quad \text{or} \quad \frac{5\pi}{3}$$

$$\therefore \theta = \frac{\pi}{3} - \frac{\pi}{6} \quad \text{or} \quad \frac{5\pi}{3} - \frac{\pi}{6}$$

$$= \frac{\pi}{6} \quad \text{or} \quad \frac{3\pi}{2}$$

Marks	Guideline
2	Correct solution
1	One correct answer or equivalent merit.
Marker's Comment	Mostly well done. Some common errors were only finding one solution.

d) If $\underset{\sim}{a} = -6\underset{\sim}{i} - 2\underset{\sim}{j}$, and $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j}$

i) Show that $\underset{\sim}{a}\cdot\underset{\sim}{b} = -10$

Solution:

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = -6\times1 + -2\times2$$
$$= -10$$

Marks	Guideline
1	For correct answer
Marker's Comment	Very well done.

ii) Hence find the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$

Solution:

$$proj_{\underset{\sim}{b}}\,\underset{\sim}{a} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}}\underset{\sim}{b}$$
$$= \frac{-10}{1+5}\left(\underset{\sim}{i} + 2\underset{\sim}{j}\right)$$
$$= -2\underset{\sim}{i} - 4\underset{\sim}{j}$$

Marks	Guideline
1	Correct solution
Marker's Comment	Mixed results. This is an area the cohort needs to revise as many did not remember the projection vector formula.

e)	Use mathematical Induction to prove that $11^{2n}+11^{n}+8$ is a multiple of 10 for $n\geq 1$.

Solution:

Let $S(n)$ be the statement that $11^{2n}+11^{n}+8$ is divisible by 10

For $n=1$ $\quad 11^{2}+11^{1}+8=140$ which is divisible by 10

Hence $S(1)$ is true

Suppose there is a k such that $S(k)$ is true.

i.e. $\quad 11^{2k}+11^{k}+8=10M$ for some $M\in\mathbb{Z}^{+}$

RTP that $11^{2(k+1)}+11^{k+1}+8$ is divisible by 10

Now

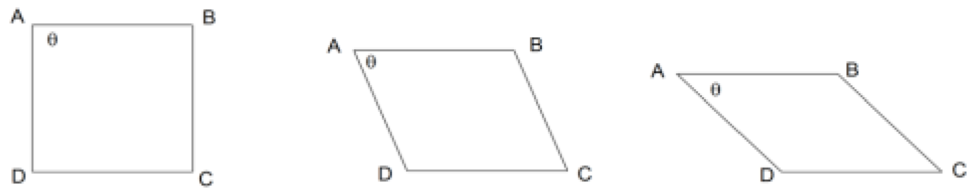
$$\begin{aligned}11^{2(k+1)}+11^{k+1}+8 &= 11^{2k}\times 11^{2}+11^{k}\times 11+8\\ &= 121\times 11^{2k}+11\times 11^{k}+8+80-80\\ &= 110\times 11^{2k}-80+11\left(11^{2k}+11^{k}+8\right)\\ &= 10\left(11\times 11^{2k}-8\right)+11\times 10M\\ &= 10\left(11\times 11^{2k}-8+11M\right)\\ &= 10N \quad \text{for some } N\in\mathbb{Z}^{+}\end{aligned}$$

Since $S(n)$ is true for $n=1$, k and $k+1$ then by the principle of mathematical induction it is true $\forall n\geq 1$.

Marks	Guideline
3	Correct solution.
2	True for n=1 and step 2 or true n=1 and some progress towards step 3.
1	True for n=1 or step 2.
Marker's Comment	Mostly well done. Common errors were not defining the pronumeral "M" or incorrectly substituting/simplifying in step 3. Another common error was not fully using k+1 in step 3. This highlights that students with incorrect responses need to revise their algebra skills.

Question 13

a)



A square $ABCD$ of side length 1 unit is “gradually” pushed over to become a rhombus. The angle at A, θ decreases at a constant rate of 0.1 radian per second?

i) At what rate is the area of the rhombus $ABCD$ decreasing when $\theta = \frac{\pi}{6}$?

Solution:

$$Area = \frac{1}{2}AD \times AB\sin\theta + \frac{1}{2}CD \times CB\sin\theta$$

$$= \sin\theta$$

$$\therefore \frac{dArea}{d\theta} = \cos\theta$$

$$\frac{dArea}{dt} = \frac{dArea}{d\theta} \times \frac{d\theta}{dt}$$

$$= \cos\theta \times 0.1$$

When $\theta = \frac{\pi}{6}$, $\frac{dArea}{dt} = \frac{\sqrt{3}}{20}$ cm^2 s^{-1}

Marks	Guideline
3	For correct answer
2	Finds area of rhombus, differentiates correctly and substitutes into function of a function rule. Or equivalent merit.
1	Uses area of a triangle formula or states function of a function rule or equivalent merit.
Marker’s Comment	

ii) At what rate is the shorter diagonal of the rhombus $ABCD$ decreasing when $\theta = \frac{\pi}{3}$?

Solution:

From the diagram the shorter diagonal is BD

Let $x = \overline{BD}$

$$\therefore x = \sqrt{1^2 + 1^2 - 2 \times 1 \times 1 \times \cos\theta}$$

$$= \sqrt{2 - 2\cos\theta}$$

$$= \sqrt{2 - 2\left(1 - 2\sin^2\left(\frac{\theta}{2}\right)\right)}$$

$$= 2\sin\left(\frac{\theta}{2}\right)$$

$$\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt}$$

$$= 2 \times \frac{1}{2}\cos\left(\frac{\theta}{2}\right) \times 0.1$$

When $\theta = \frac{\pi}{3}$, $\frac{dx}{dt} = \frac{\sqrt{3}}{20}$ cms^{-1}

Marks	Guideline
3	Correct solution
2	Correct solution for their shorter diagonal or correct shorter diagonal and its derivative or equivalent merit.
1	Correct expression for shorter diagonal.
Marker’s Comment	

b)	Show that the solution to the differential equation $\frac{dy}{dx}=1+\frac{2y}{3}$, given $y=-1$ when $x=2$ can be written in the form $y=\frac{e^{\frac{2(x-2)}{3}}-3}{2}$.

Solution:

$$\frac{dy}{dx}=1+\frac{2y}{3}$$

$$=\frac{3+2y}{3}$$

$$\frac{dx}{dy}=\frac{3}{3+2y}$$

$$=\frac{3}{2}\times\frac{2}{3+2y}$$

$$\therefore x=\frac{3}{2}\ln|3+2y|+c$$

At $x=2, y=-1 \Rightarrow 2=\frac{3}{2}\ln 1+c \Rightarrow c=2$

$$\therefore x=\frac{3}{2}\ln|3+2y|+2$$

$$\therefore \ln|3+2y|=\frac{2(x-2)}{3}$$

$$\therefore 3+2y=e^{2(x-2)/3}$$

$$\therefore y=\frac{e^{2(x-2)/3}-3}{2}$$

Marks	Guideline
4	Correct solution
3	Correct indefinite integral and value of *c* or equivalent merit.
2	Correct indefinite integral or equivalent merit.
1	Re-arranges equation into a usable form or calculates their value of *c* or equivalent merit.
Marker's Comment	

c)	A rabbit population grows according to the logistic equation $\frac{dP}{dt}=0.2P\left(1-\frac{P}{10000}\right)$, where P is the number of rabbits after t months. The initial population is 1000.
i)	Show that $\frac{10000}{P(10000-P)}=\frac{1}{P}+\frac{1}{10000-P}$

Solution:

$$\text{RHS} =\frac{1}{P}+\frac{1}{10000-P}$$
$$=\frac{10000-P+P}{P(10000-P)}$$
$$=\frac{10000}{P(10000-P)}$$
$$=\text{LHS}$$

Marks	Guideline
1	Correct solution
Marker's Comment	

ii)	Find the population of rabbits after seven months.

Solution:

$$\frac{dP}{dt}=0.2P\left(1-\frac{P}{10000}\right)$$
$$5\frac{dP}{dt}=P\left(\frac{10000-P}{10000}\right)$$
$$\frac{1}{5}\frac{dt}{dP}=\frac{10000}{P(10000-P)}$$
$$=\frac{1}{P}+\frac{1}{10000-P}$$
$$\therefore \frac{1}{5}t =\ln P-\ln|10000-P|+c$$

At $t=0,\ P=1000 \Rightarrow\ 0=\ln 1000-\ln 9000+c\ \Rightarrow c=\ln 9$

$$\therefore \frac{t}{5} =\ln P-\ln(10000-P)+\ln 9$$
$$=\ln\left(\frac{9P}{10000-P}\right)$$

At $t=7$, $\frac{7}{5}=\ln\left(\frac{9P}{10000-P}\right)$

$$\therefore \frac{9P}{10000-P}=e^{7/5}$$
$$\therefore 9P+Pe^{7/5}=10000e^{7/5}$$
$$\therefore P =\frac{10000e^{7/5}}{9+e^{7/5}}$$
$$\approx 3106.19$$

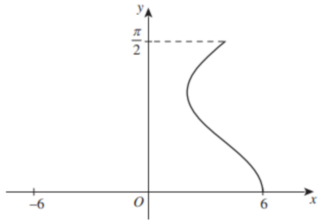
The population after seven months is 3106.

Marks	Guideline
4	For correct answer
3	Uses value of *t* to find correct exponential equation or equivalent merit.
2	Correct indefinite integral and value of *c* or equivalent merit.
1	Re-arranges equation into a usable form or calculates their value of *c* or equivalent merit.
Marker's Comment	

Question 14

a) The diagram shows the region bounded by the y-axis and $x = 4\cos(3y) + 2$, where $0 \le y \le \frac{\pi}{2}$.

The region is rotated about the y-axis to form a solid.



Find the exact volume, V, of the solid formed.

Solution:

$$V = \pi \int_0^{\pi/2} x^2 \, dy$$

$$= \pi \int_0^{\pi/2} \left(16\cos^2(3y) + 16\cos(3y) + 4\right) dy$$

$$= \pi \int_0^{\pi/2} \left(8 + 8\cos(6y) + 16\cos(3y) + 4\right) dy$$

$$= \pi \left[12y + \frac{8}{6}\sin(6y) + \frac{16}{3}\sin(3y)\right]_0^{\pi/2}$$

$$= \pi \left[6\pi + \frac{4}{3}\sin 3\pi + \frac{16}{3}\sin\frac{3\pi}{2} - 0\right]$$

$$= \pi \left(\frac{18\pi - 16}{3}\right) \text{ units}^3$$

Marks	Guideline
3	For correct answer
2	Correct integral or correct solution from their integral involving a $\cos^2$ term. Or equivalent merit.
1	Correct expression for x^2 or correct substitution for their $\cos^2$ expression or correct solution for their integral or equivalent merit.
Marker's Comment	Generally well done. Some students missed the $16\cos(3y)$ or changed the $16\cos^2(3y)$ incorrectly.

b) A particle, A, is projected from the origin with an initial velocity of $16\underset{\sim}{i} + 30\underset{\sim}{j}$ ms^{-1}.

At the same time, particle B is projected towards the origin from a point that is 60 m to the right of the origin and 25 m above the origin with an initial velocity of $-8\underset{\sim}{i} + 20\underset{\sim}{j}$ ms^{-1}.

Let the acceleration due to gravity be 9.8 ms^{-2}.

The position vector of particle A at time t seconds is given by $A(t) = 16t\underset{\sim}{i} + \left(30t - 4.9t^2\right)\underset{\sim}{j}$.

i) Find the position vector of particle B (i.e. $B(t)$) at time t seconds.

Solution:

For particle B

$\ddot{y} = -g$ $\qquad$ $\ddot{x} = 0$

$\dot{y} = -gt + 20$ $\qquad$ $\dot{x} = -8$

$y = -\frac{g}{2}t^2 + 20t + 25$ $\qquad$ $x = -8t + 60$

$\therefore B(t) = (-8t + 60)\underset{\sim}{i} + \left(25 + 20t - 4.9t^2\right)\underset{\sim}{j}$

Marks	Guideline
3	Correct solution
2	One correct component and progress towards the other component.
1	One correct component.
Marker's Comment	Well done, although some students had $x = -8t$ and $y = -\frac{g}{2}t^2 + 20t$

ii) Find the time and point at which the two particles collide.

Solution:

The particles collide when $A(t)=B(t)$

$\therefore 16t\underset{\sim}{i}+\left(30t-4.9t^2\right)\underset{\sim}{j}=(-8t+60)\underset{\sim}{i}+\left(25+20t-4.9t^2\right)\underset{\sim}{j}$

Comparing $\underset{\sim}{i}$ components: $-8t+60=16t \Rightarrow t=2.5$

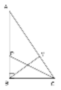
$$A(2.5)=16\times 2.5\underset{\sim}{i}+\left(30\times 2.5-4.9(2.5)^2\right)\underset{\sim}{j}$$

$$=40\underset{\sim}{i}+\frac{355}{8}\underset{\sim}{j}$$

The particles collide after 2.5 s at a point 40 m to the right and 44.375 m above the origin

Marks	Guideline
3	Correct solution
2	Finds the time at which the two particles collide or equivalent merit.
1	Recognises we need to make $A(t)=B(t)$.
Marker's Comment	Well done

c) In the figure $\angle ABC=90°$, $PA=2\times BP$, $AM=2\times MC$.



Let $\overrightarrow{BA}=\underset{\sim}{v}$ and $\overrightarrow{BC}=\underset{\sim}{u}$.

i) $\overrightarrow{BM}$ and $\overrightarrow{CP}$ in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

Solution:

Now $\overrightarrow{CA}=\overrightarrow{BA}-\overrightarrow{BC}=\underset{\sim}{v}-\underset{\sim}{u}$

$\therefore \overrightarrow{CM}=\frac{1}{3}(\underset{\sim}{v}-\underset{\sim}{u})$ and $\therefore \overrightarrow{BP}=\frac{1}{3}\underset{\sim}{v}$

$$\overrightarrow{BM}=\overrightarrow{BC}+\overrightarrow{CM}=\underset{\sim}{u}+\frac{1}{3}(\underset{\sim}{v}-\underset{\sim}{u})=\frac{2}{3}\underset{\sim}{u}+\frac{1}{3}\underset{\sim}{v}$$

$$\overrightarrow{CP}=\overrightarrow{CB}+\overrightarrow{BP}=-\underset{\sim}{u}+\frac{1}{3}\underset{\sim}{v}$$

Marks	Guideline
3	Correct solution
2	Finds either *BM* or *CP*.
1	Finds *CM* or *BP*.
Marker's Comment	Many, many students had the vectors in the wrong direction e.g., $\overrightarrow{CP}=\overrightarrow{BC}+\overrightarrow{BP}=\underset{\sim}{u}+\frac{\underset{\sim}{v}}{3}$ Some had $\frac{1}{2}$ instead of $\frac{1}{3}$

ii) If $BM\perp CP$, find $\frac{\lvert BA\rvert}{\lvert BC\rvert}$

Solution:

Now $\overrightarrow{BM}\cdot\overrightarrow{CP}=0 \Rightarrow \left(\frac{2}{3}\underset{\sim}{u}+\frac{1}{3}\underset{\sim}{v}\right)\cdot\left(-\underset{\sim}{u}+\frac{1}{3}\underset{\sim}{v}\right)=0$

$\therefore -\frac{2}{3}(\underset{\sim}{u}\cdot\underset{\sim}{u})+\frac{2}{9}(\underset{\sim}{u}\cdot\underset{\sim}{v})-\frac{1}{3}(\underset{\sim}{v}\cdot\underset{\sim}{u})+\frac{1}{9}(\underset{\sim}{v}\cdot\underset{\sim}{v})=0$

$\therefore \frac{1}{9}(\underset{\sim}{v}\cdot\underset{\sim}{v})-\frac{2}{3}(\underset{\sim}{u}\cdot\underset{\sim}{u})=0$ as $(\underset{\sim}{u}\cdot\underset{\sim}{v}=0)$

$\therefore \frac{\underset{\sim}{v}\cdot\underset{\sim}{v}}{\underset{\sim}{u}\cdot\underset{\sim}{u}}=6$

$\therefore \frac{\lvert\underset{\sim}{v}\rvert^2}{\lvert\underset{\sim}{u}\rvert^2}=6 \Rightarrow \frac{\lvert BA\rvert}{\lvert BC\rvert}=\frac{\lvert\underset{\sim}{v}\rvert}{\lvert\underset{\sim}{u}\rvert}=\sqrt{6}$

Marks	Guideline
3	For correct answer
2	Uses $u.v=0$ to simplify equation or equivalent merit.
1	Uses $BM.CP=0$.
Marker's Comment	Generally well done, although there were some issues with expanding the dot product.